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Monte Carlo Sampling Processes and Incentive Compatible Allocations in Large Economies

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Monte Carlo Sampling Processes and Incentive Compatible Allocations in Large Economies

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Abstract

Monte Carlo simulation is used in [13] to characterize a standard stochastic framework involving a continuum of random variables that are conditionally independent given macro shocks. This paper presents some general properties of such Monte Carlo sampling processes, including their one-way Fubini extension and regular conditional independence. In addition to the almost sure convergence of Monte Carlo simulation considered in [13], here we also consider norm convergence when the random variables are square integrable. This leads to a necessary and sufficient condition for the classical law of large numbers to hold in a general Hilbert space. Applying this analysis to large economies with asymmetric information shows that the conflict between incentive compatibility and Pareto efficiency is resolved asymptotically for almost all sampling economies, corresponding to some results in [21] and [24].

KEYWORDS: Law of large numbers, Monte Carlo sampling process, one-way Fubini property, Hilbert space, incentive compatibility, asymmetric information, Pareto efficiency.

JEL Classification: C65, D51, D61, D82

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Contents

1 Introduction **2**

2 Basic formulation **3**

2.1 Monte Carlo sampling processes 3

2.2 One-way Fubini property 4

2.3 Regular conditional independence 4

3 Properties of Monte Carlo sampling processes **5**

4 Characterizing a sharp law of large numbers **5**

5 Allocations in large economies with asymmetric information **8**

5.1 The information structure 8

5.2 A state contingent large economy 10

5.3 Monte Carlo sampling economies 11

5.4 Asymptotically feasible and Pareto efficient incentive compatible allocations . . 11

6 Appendix **13**

6.1 Iteratively complete products 13

6.2 Proof of Theorem 1 14

6.3 Proofs of the results in Section 4 15

6.4 Proof of Theorem 3 21

1 Introduction

Following the early writings [20, 4], macroeconomists have made widespread use of a model of an economy with many agents who face individual random shocks. These shocks are typically modelled as a continuum of random variables that are conditionally independent given common macro level shocks. Proposition 4 in [13], however, shows that in this framework, the joint measurability condition that is usually imposed on a stochastic process can be satisfied only if there is essentially no idiosyncratic risk at all. The approach of Monte Carlo simulation is then used in [13] to characterize when, even in the absence of the usual joint measurability assumption, the standard stochastic framework for many heterogeneous agents facing individual uncertainty may still be valid. This paper provides a systematic study of the underlying Monte Carlo sampling processes. We also present an application involving allocations in large exchange economies with many asymmetrically informed consumers. In particular, we show how Monte Carlo sampling helps resolve the conflict between incentive compatibility and Pareto efficiency vanishes in the limit as the number of agents tends to infinity.

Let

$$I \times \Omega \ni (i, \omega) \mapsto g_i(\omega) \in X$$

be a process with a continuum of random variables, indexed by members i of an atomless probability space $(I, \mathcal{I}, \lambda)$, all defined on the same sample probability space $(\Omega, \mathcal{F}, P)$, and taking values in a Polish space X. Let I^∞ and X^∞ denote the Cartesian product of infinitely many copies of the sets I and X respectively, with typical members $i^\infty = \langle i_k \rangle_{k=1}^\infty$ and $x^\infty = \langle x_k \rangle_{k=1}^\infty$. Then the corresponding Monte Carlo sampling process G is a mapping

$$I^\infty \times \Omega \ni (i^\infty, \omega) \mapsto G(i^\infty, \omega) = \langle g(i_k, \omega) \rangle_{k=1}^\infty \in X^\infty \tag{1}$$

When the process g has a stochastic macro structure, as defined in Section 2.3, Theorem 1 shows that so does the Monte Carlo sampling process G. In this case, the process G also has the property of admitting a "one-way Fubini extension" that makes G jointly measurable with respect to an extension of the usual product σ-algebra.

The Monte Carlo simulation approach in [13] uses the almost sure convergence of the sample averages. For a square-integrable process, we also consider here the case of norm convergence. Based on the iterative extension of an infinite product measure introduced in [12], we formulate a "sharp" law of large numbers, requiring norm convergence of sample averages only for all sequences outside an iteratively null set, rather than a smaller classical null set. We prove that a process with square-integrable random variables satisfies this sharp law if and only if it is both Gel′fand-integrable and norm integrably bounded in the Hilbert space of square integrable random variables. In other words, this result characterizes those processes with square-integrable random variables whose average conditional expectation, given the macro states, can be estimated using Monte Carlo simulation.

For allocations in a finite-agent asymmetric information economy, it is well known that there is a conflict between incentive compatibility and Pareto efficiency (see, for example, Example 0.1 on p. vi of [10]). The papers [21] and [24] show the (approximate) consistency of incentive compatibility and efficiency by working with, respectively: (i) a large but finite set of agents; (ii) a continuum of agents. In this paper, in the setting of a sequence of economies that result from Monte Carlo sampling, we show that the conflict between incentive compatibility and Pareto efficiency is resolved asymptotically for almost all infinite sequences of economies. This corresponds to the asymptotic result for replica economies in [21], and the exact result in [24] when private signals are generated by a process that is jointly measurable in a two-way Fubini extension.

The rest of the paper is organized as follows. Section 2 includes some basic definitions. Some general properties of the Monte Carlo sampling processes are presented in Section 3. Section 4 provides a necessary and sufficient condition for the classical law of large numbers to hold in a general Hilbert space. As an illustrative application, Section 5 shows that in a Monte Carlo sampled sequence of economies with asymmetric information, incentive compatibility and Pareto efficiency are asymptotically consistent. Additional definitions and all the proofs are given in the Appendix.

2 Basic formulation

2.1 Monte Carlo sampling processes

We model a continuum $\langle g_i \rangle_{i \in I}$ of random variables indexed by $i \in I$ as a *process* $g : I \times \Omega \to X$ where:

1. $(I, \mathcal{I}, \lambda)$ is an atomless probability space, often the Lebesgue unit interval, whose typical member is an *index* i that identifies one particular economic agent;
2. $(\Omega, \mathcal{F}, P)$ is a probability space that represents the overall risk in the process;[1]
3. $(X, \mathcal{B})$ is a Polish space with its Borel σ-algebra;
4. each indexed function $g_i : \Omega \to X$ is measurable, so a random variable;
5. for each fixed $B \in \mathcal{B}$, the mapping $I \ni i \mapsto (P \circ g_i^{-1})(B)$ from I to the probability that $g_i(\omega) \in B$ is measurable.

[1]We follow the convention that a probability space is assumed to be countably additive, as well as complete in the sense that the σ-algebra $\mathcal{F}$ includes all subsets of every P-null set.

A (Monte Carlo) sample of the indices $i \in I$ is a countable collection $i^\infty = \langle i_k \rangle_{k=1}^\infty$ drawn from the iteratively completed infinite product probability space $(I^\infty, \bar{\mathcal{I}}^\infty, \bar{\lambda}^\infty)$ defined in Section 6.1. This space was introduced in [12] as the usual infinite product probability space $(I^\infty, \mathcal{I}^\infty, \lambda^\infty)$ extended so that the σ-algebra $\bar{\mathcal{I}}^\infty$ includes iteratively null sets.

Corresponding to each (Monte Carlo) sample $i^\infty = \langle i_k \rangle_{k=1}^\infty$ of indices is a countable sequence $\langle g_{i_k} \rangle_{k=1}^\infty$ of random variables. This constitutes a (Monte Carlo) sample from the continuum of random variables $\Omega \ni \omega \mapsto g_i(\omega) \in X$ as i varies over I. This sample, with $i^\infty \in I^\infty$ fixed, can be regarded as part of one meta or (Monte Carlo) *sampling process* G defined by (1).

2.2 One-way Fubini property

The following definition was introduced in [11].

Definition 1 *A probability space* $(I \times \Omega, \mathcal{W}, Q)$ extends *the usual product probability space* $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \times P)$ *provided that* $\mathcal{W} \supseteq \mathcal{I} \otimes \mathcal{F}$, *with* $Q(E) = (\lambda \times P)(E)$ *for all* $E \in \mathcal{I} \otimes \mathcal{F}$.

The extended space $(I \times \Omega, \mathcal{W}, Q)$ *is a* one-way Fubini extension *of the product probability space* $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \times P)$ *provided that, given any* Q*-integrable function* $I \times \Omega \ni (i, \omega) \mapsto f(i, \omega) \in \mathbb{R}$:

(i) for λ*-almost all* $i \in I$, *the random variable* $\omega \mapsto f_i(\omega)$ *is integrable on* $(\Omega, \mathcal{F}, P)$;

(ii) the function $i \mapsto \int_\Omega f_i \, dP$ *is integrable on* $(I, \mathcal{I}, \lambda)$, *with* $\int_{I \times \Omega} f \, dQ = \int_I \left(\int_\Omega f_i \, dP \right) d\lambda$.

A process $g : I \times \Omega \to X$ *is said to satisfy the* one-way Fubini property *if there is a one-way Fubini extension* $(I \times \Omega, \mathcal{W}, Q)$ *such that* g *is* $\mathcal{W}$*-measurable.*

2.3 Regular conditional independence

A σ-algebra $\mathcal{C}$ on Ω is said to be *countably generated* if there exists a countable family $\{ C_n \}_{n=1}^\infty$ of subsets of Ω that generates $\mathcal{C}$. Given a complete probability space $(\Omega, \mathcal{F}, P)$, a sub-σ-algebra $\mathcal{C}$ of $\mathcal{F}$ is said to be *countably generated* if it is the *strong completion* of a countably generated σ-algebra $\mathcal{C}'$, in the sense that

$$\mathcal{C} = \{ A \in \mathcal{F} \mid \exists A' \in \mathcal{C}' : P(A \triangle A') = 0 \}$$

Definition 2 *Let* g *be a process from* $I \times \Omega$ *to the Polish space* X *with its Borel* σ*-algebra* $\mathcal{B}$. *Let* $\mathcal{C}$ *be a countably generated sub-*σ*-algebra of* $\mathcal{F}$ *in the complete probability space* $(\Omega, \mathcal{I}, P)$. *Let* $\mathcal{M}(X)$ *denote the space of probability measures on the space* $(X, \mathcal{B})$.

1. *Two random variables* ϕ *and* ψ *that map* $(\Omega, \mathcal{F}, P)$ *to* X *are said to be* conditionally independent *given* $\mathcal{C}$ *if, for any Borel sets* $B_1, B_2 \in \mathcal{B}$, *the conditional probabilities satisfy*

$$P(\phi^{-1}(B_1) \cap \psi^{-1}(B_2) | \mathcal{C}) = P(\phi^{-1}(B_1) | \mathcal{C}) \, P(\psi^{-1}(B_2) | \mathcal{C}) \tag{2}$$

2. *The process g is said to be* essentially pairwise conditionally independent *given $\mathcal{C}$ if, for λ-a.e. $i_1 \in I$, the random variables g_{i_1} and g_{i_2} are conditionally independent given $\mathcal{C}$ for λ-a.e. $i_2 \in I$.*

3. *An $\mathcal{I} \otimes \mathcal{C}$-measurable mapping μ from $I \times \Omega$ to $\mathcal{M}(X)$ is said to be an* essentially regular conditional distribution process *of g given $\mathcal{C}$ if, for λ-a.e. $i \in I$, the $\mathcal{C}$-measurable mapping $\omega \mapsto \mu_{i\omega}$ is a regular conditional distribution $P(g_i^{-1}|\mathcal{C})$ of the random variable g_i.*

4. *The process g is said to be* regularly conditionally independent *given $\mathcal{C}$ if g is essentially pairwise conditionally independent given $\mathcal{C}$, and also g admits* a stochastic macro structure *$(\mathcal{C}, \mu)$ in the form of an essentially regular conditional distribution process μ given $\mathcal{C}$.*

3 Properties of Monte Carlo sampling processes

Let g be a process from $I \times \Omega$ to X and $\mathcal{C}$ a countably generated sub-σ-algebra of $\mathcal{F}$ in $(\Omega, \mathcal{F}, P)$. Suppose that g is essentially pairwise conditionally independent given $\mathcal{C}$ and admits an essentially regular conditional distribution process $I \times \Omega \ni (i, \omega) \mapsto \mu_{i\omega} \in \mathcal{M}(X)$ given $\mathcal{C}$. By Theorem 1 in [14], there exists a one-way Fubini extension $(I \times \Omega, \mathcal{W}, Q)$ such that the process g is $\mathcal{W}$-measurable.

The following theorem shows that the Monte Carlo sampling process G defined by (1) has the one-way Fubini property. It also shows that G satisfies regular conditional independence given $\mathcal{C}$, and identifies its regular conditional distribution process.

Let $\mathcal{M}(X^\infty)$ denote the set of probability measures on the infinite product measurable space $(X^\infty, \mathcal{B}^\infty)$.

Theorem 1 *Let $G : I^\infty \times \Omega \to X^\infty$ be a Monte Carlo sampling process of g. Then there exists a one-way Fubini extension $(I^\infty \times \Omega, \tilde{\mathcal{W}}, \tilde{Q})$ of $(I^\infty \times \Omega, \bar{\mathcal{I}}^\infty \otimes \mathcal{F}, \bar{\lambda}^\infty \otimes P)$ such that G is $\tilde{\mathcal{W}}$-measurable. In addition, G is essentially pairwise conditionally independent given $\mathcal{C}$. It also admits the essentially regular conditional distribution process $\bar{\mu}$ defined by*

$$I^\infty \times \Omega \ni (i^\infty, \omega) \mapsto \bar{\mu}(i^\infty, \omega) := \prod_{k=1}^{\infty} \mu(i_k, \omega) \in \mathcal{M}(X^\infty) \tag{3}$$

4 Characterizing a sharp law of large numbers

Let g be a process from $I \times \Omega$ to X as in Section 3, and let $h : I \times X \to \mathbb{R}$ be an $\mathcal{I} \otimes \mathcal{B}$-measurable function with $\int_I \left[\int_\Omega h_i^2(g_i(\omega))\, dP\right] d\lambda < \infty$. Then Lemma 11 in [13] says that for $\bar{\lambda}^\infty$-a.e. $i^\infty \in I^\infty$, one has

$$\frac{1}{n} \sum_{k=1}^{n} h(i_k, g(i_k, \omega)) \to \int_I \left[\int_X h(i, x)\, d\mu_{i\omega}\right] d\lambda \text{ for } P\text{-almost all } \omega \in \Omega$$

Under the framework of one-way Fubini extension, we have the following corollary.

Corollary 1 *Let $G : I^\infty \times \Omega \to X^\infty$ be the Monte Carlo sampling process of g. Suppose that $(I^\infty \times \Omega, \tilde{\mathcal{W}}, \tilde{Q})$ is a one-way Fubini extension of $(I^\infty \times \Omega, \bar{\mathcal{I}}^\infty \otimes \mathcal{F}, \bar{\lambda}^\infty \otimes P)$ such that G is $\tilde{\mathcal{W}}$-measurable. Then for $\tilde{Q}$-almost all $(i^\infty, \omega) \in I^\infty \times \Omega$, one has*

$$\frac{1}{n}\sum_{k=1}^{n} h(i_k, g(i_k, \omega)) \to \int_I \left[\int_X h(i,x)\, d\mu_{i\omega}\right] d\lambda$$

Let $L^2(P)$ be the space of real-valued square integrable functions on $(\Omega, \mathcal{F}, P)$, made into a Hilbert space by defining, for any pair $\varphi, \psi \in L^2(P)$, the standard inner product $\langle \varphi, \psi \rangle := \int_\Omega \varphi(\omega)\psi(\omega)\, dP$. For each fixed $i \in I$, define the random variable $f(i)(\cdot)$ so that

$$\Omega \ni \omega \mapsto f(i)(\omega) = h\left(i, g(i,\omega)\right) \in \mathbb{R} \tag{4}$$

The assumption that $\int_I \left[\int_\Omega h_i^2(g_i(\omega))\, dP\right] d\lambda < \infty$ implies that for λ-almost all $i \in I$, the random variable $f(i)$ is an element in the Hilbert space $L^2(P)$. Corollary 1 indicates that the sample average $\frac{1}{n}\sum_{k=1}^{n} f(i_k)(\omega)$ converges $\tilde{Q}$-almost surely to $\int_I \left[\int_X h(i,x)\, d\mu_{i\omega}\right] d\lambda$. Since the function $I \ni i \mapsto f(i)$ takes values in the Hilbert space $L^2(P)$, a natural question is whether one can obtain a similar result for convergence in the norm of $L^2(P)$. This is answered in the following proposition.

Proposition 1 *For $\bar{\lambda}^\infty$-almost all $i^\infty \in I^\infty$, one has*

$$\left\| \frac{1}{n}\sum_{k=1}^{n} f(i_k) - \int_I \left[\int_X h(i,x)\, d\mu_{i\omega}\right] d\lambda \right\| \to 0$$

where the random variable $\omega \mapsto \int_I \left[\int_X h(i,x)\, d\mu_{i\omega}\right] d\lambda$ is in $L^2(P)$, and $\|\cdot\|$ denotes the standard norm on the Hilbert space $L^2(P)$.

This result can be viewed as the classical law of large numbers for a sequence of random variables taking values in the Hilbert space $L^2(P)$. One may wonder whether such a result can be extended to other Hilbert spaces, or to Banach spaces more generally.

Let $\mathbb{B}$ be a Banach space, with norm $\|\cdot\|$ and norm dual $\mathbb{B}'$. Given any $b \in \mathbb{B}$ and $b' \in \mathbb{B}'$, let $\langle b, b' \rangle$ denote the real value of the continuous linear mapping b' evaluated at b. In the case when $\mathbb{B}$ is a Hilbert space, we shall denote it by $\mathbb{H}$. Then, of course, $\langle b, b' \rangle$ can be regarded as the inner product.

Henceforth we use the respective abbreviations LLN and SLLN for the law of large numbers, and the sharp law of large numbers.

Definition 3 *Let f be a function from $(I, \mathcal{I}, \lambda)$ to a Banach space $\mathbb{B}$.*

1. *The function f is said to satisfy LLN (resp., SLLN) if there exists $a \in \mathbb{B}$ such that $\|a - \frac{1}{n}\sum_{k=1}^{n} f(i_k)\| \to 0$ λ^∞-a.s. (resp., $\bar{\lambda}^\infty$-a.s.). Let* LLN($\mathbb{B}$) *(resp.,* SLLN($\mathbb{B}$) *denote the (linear) space of all functions from I to $\mathbb{B}$ that satisfy LLN (resp., SLLN).*

2. *The function f is said to be **Gel′fand integrable** if there exists a vector $b \in \mathbb{B}$ called the **Gel′fand integral** of f such that, for all $b' \in \mathbb{B}'$, the real-valued function $i \mapsto \langle f(\cdot), b' \rangle$ on I is λ-integrable, with $\int_I \langle f(i), b' \rangle d\lambda = \langle b, b' \rangle$.*[2]

3. *A function $f^* : I \to \mathbb{R}_+$ **norm dominates** $f : I \to \mathbb{B}$ if $\|f(i)\| \le f^*(i)$ for λ-a.e. $i \in I$. The function f is said to be **norm integrably bounded** if there exists a λ-integrable function $f^* : I \to \mathbb{R}_+$ that norm dominates f.*

From now on, let $\mathcal{L}(\lambda, \mathbb{B})$ denote the (linear) space of all functions f from $(I, \mathcal{I}, \lambda)$ to $\mathbb{B}$ that are both Gel′fand integrable and norm integrably bounded. The following Proposition is well known in the literature on random variables with values in a Banach space. For Part (1), see for example [17, Theorem 2.4]. Part (2) is taken from [8, Theorem 3.1].

Proposition 2 *(1)* LLN($\mathbb{B}$) $\subseteq \mathcal{L}(\lambda, \mathbb{B})$ *for any Banach space $\mathbb{B}$.*

(2) Suppose that I is a Polish space, that $\mathcal{I}$ is its Borel σ-algebra, and that λ is an atomless probability measure. There is a Hilbert space $\mathbb{H}$ such that $\mathcal{L}(\lambda, \mathbb{H})$ is not equal to LLN($\mathbb{H}$).

Part (1) of Proposition 2 says that a necessary condition for f to satisfy the usual LLN is that f must be both Gel′fand integrable and norm integrably bounded. On the other hand, Part (2) of Proposition 2 shows that these two conditions are not sufficient for the LLN to hold, even for the special case of a Hilbert space. It means that LLN($\mathbb{B}$) is in general a proper subset of $\mathcal{L}(\lambda, \mathbb{B})$.

The following theorem shows that when the product measure λ^∞ is extended to its iterated completion $\bar{\lambda}^\infty$, not only does the strengthened inclusion SLLN($\mathbb{B}$) $\subseteq \mathcal{L}(\lambda, \mathbb{B})$ hold for a general Banach space $\mathbb{B}$, but it becomes an equality in the Hilbert space case. This equality provides a very general characterization of the functions from $(I, \mathcal{I}, \lambda)$ to a general Hilbert space $\mathbb{H}$ that satisfy our sharp law of large numbers in the iterated completion of the product probability space. Moreover, an obvious corollary of our results is that LLN($\mathbb{B}$) is in general a proper subset of SLLN($\mathbb{B}$), even when $\mathbb{B}$ is a Hilbert space.

[2]This follows the terminology of [8] and [17]. When the Gel′fand integral of $1_S f$ is defined for every $S \in \mathcal{I}$, this is often called the *Pettis integral* — see, for example, [1] and [7, p. 53]. Note that 1_S is the indicator function of the set S.

Theorem 2 (Sharp law of large numbers) *If f is any function from $(I,\mathcal{I},\lambda)$ to a Banach space $\mathbb{B}$ for which SLLN is satisfied, then $f \in \mathcal{L}(\lambda,\mathbb{B})$; that is,* $\mathrm{SLLN}(\mathbb{B}) \subseteq \mathcal{L}(\lambda,\mathbb{B})$. *More importantly, if $\mathbb{B}$ is a Hilbert space $\mathbb{H}$, then f satisfies SLLN if and only if f is Gel′fand integrable and norm integrably bounded. That is,* $\mathrm{SLLN}(\mathbb{H}) = \mathcal{L}(\lambda,\mathbb{H})$.

Consider the function f from $(I,\mathcal{I},\lambda)$ to the Hilbert space $L^2(P)$ which is defined by (4). Because $\|f(i)\| = \left(\int_\Omega h_i^2(g_i(\omega))\,dP\right)^{\frac{1}{2}}$ and $\int_I \left[\int_\Omega h_i^2(g_i(\omega))\,dP\right]\,d\lambda < \infty$, the function f is obviously norm integrably bounded. The following claim, which says that such a function f is also Gel′fand integrable, indicates why Proposition 1 is a special case of Theorem 2.

Claim 1 *Let f be the function from $(I,\mathcal{I},\lambda)$ to $L^2(P)$ defined by $f(i)(\omega) = h\left(i, g(i,\omega)\right)$ in the paragraph above Proposition 1. Then, f is Gel′fand integrable.*

5 Allocations in large economies with asymmetric information

5.1 The information structure

We use the same information structure as that set out in [24]. Suppose that the fixed atomless probability space $(I,\mathcal{I},\lambda)$ represents the space of economic agents. Let $S = \{s_1, s_2, \ldots, s_K\}$ denote the finite set of **true states** of nature (with power set denoted by $\mathcal{S}$). We assume that these are not known by any agent. Let $T^0 = \{q_1, q_2, \ldots, q_L\}$ denote the space of all possible signals (or types) for each individual agent. We consider the measurable space $(T,\mathcal{T})$ of private signal or type profiles for all the agents $i \in I$. Thus, T is a subset of $(T^0)^I$, the space of all functions from I to T^0.[3] For each agent $i \in I$, the function value $t(i)$ (also denoted by t_i) is agent i's **private signal**, whereas t_{-i} is the restriction of the signal profile t to the set $I \setminus \{i\}$ of agents different from i; let T_{-i} denote the set of all such t_{-i}. For simplicity, we assume that $(T,\mathcal{T})$ has a rich enough product structure so that T is a product of T_{-i} and T^0, whereas $\mathcal{T}$ is the product σ-algebra of the power set $\mathcal{T}^0$ on T^0 with a σ-algebra $\mathcal{T}_{-i}$ on T_{-i}. Given any $t \in T$ and $t_i' \in T^0$, we shall adopt the usual notation (t_{-i}, t_i') to denote the signal profile whose value is t_i' for agent i, but the same as t_j for all other agents $j \in I \setminus \{i\}$.

To represent all the **uncertainty** about the true states as well as the agents' signals, we consider the probability space $(\Omega,\mathcal{F},P)$ where $(\Omega,\mathcal{F})$ is the product measurable space $(S \times T, \mathcal{S} \otimes \mathcal{T})$. Let P^S and P^T be the marginal probability measures of P on $(S,\mathcal{S})$ and $(T,\mathcal{T})$ respectively. For each $i \in I$, let $\tilde{s}$ and $\tilde{t}_i$ denote the projection mappings from Ω to S and to T^0 respectively, with $\tilde{t}_i(s,t) = t_i$.[4] After excluding any P^S-null state, we assume without loss of

[3] The standard literature usually assumes that different agents have different sets of possible signals, all of which occur with positive probability. For notational simplicity, we choose to work instead with a common set T^0 of possible signals, but allow some of these to have zero probability for some agents. There is no loss of generality in this latter approach.

[4] Because $\Omega = S \times T$, the mapping $\tilde{t}_i$ can also be viewed as a projection from T to T^0.

generality that each true state $s \in S$ is *non-null* in the sense that $\pi_s := P^S(\{s\}) > 0$; let P_s^T be the conditional probability measure on $(T, \mathcal{T})$ given that the random variable $\tilde{s}$ takes value s. Thus, for each $B \in \mathcal{T}$, one has $P_s^T(B) = P(\{s\} \times B)/\pi_s$. It is obvious that $P^T = \sum_{s \in S} \pi_s P_s^T$. Note that in the literature the conditional probability measure P_s^T is often denoted as $P(\cdot|s)$.

For each fixed $t \in T$, define also the conditional probability measure $P^S(\cdot|t)$ on S so that for each fixed $s \in S$, the mapping $T \ni t \mapsto P^S(\{s\}|t)$ is $\mathcal{T}$-measurable, with $P(\{s\} \times B) = \int_B P^S(\{s\}|t)\, dP^T$ for each $B \in \mathcal{T}$. Let $T \ni t \mapsto p_s(t) \in \mathbb{R}_+$ be the density function of P_s^T with respect to P^T; it is easy to see that $P^S(\{s\}|t) = \pi_s\, p_s(t)$ for P^T-almost all $t \in T$.

For each $i \in I$, let τ_i denote the marginal signal distribution of agent i on the space T^0; it is defined so that for all $q \in T^0$, the probability $P(\tilde{t}_i = q)$ equals $\tau_i(\{q\})$. Let $P^{S \times T_{-i}}(\cdot|t_i)$ denote the conditional probability measure on the product measurable space $(S \times T_{-i}, \mathcal{S} \otimes \mathcal{T}_{-i})$ given that agent i's signal is $t_i \in T^0$. For any $t_i \in T^0$ with marginal probability $\tau_i(\{t_i\}) > 0$, it is clear that for any $E \in \mathcal{S} \otimes \mathcal{T}_{-i}$, one has $P^{S \times T_{-i}}(E|t_i) = P(E \times \{t_i\})/\tau_i(\{t_i\})$. Denote by $\mathcal{C}$ the completed sub-σ-algebra of $\mathcal{F} = \mathcal{S} \otimes \mathcal{T}$ on $\Omega = S \times T$ that is generated by the union of the finite family $\{\{s\} \times T : s \in S\}$ with the set of all P-null subsets of $S \times T$.

Let f denote the **private signal process** from $I \times \Omega$ to the finite type space T^0 defined so that $f(i, \omega) = \tilde{t}_i(\omega)$. Typically, for each $\omega \in \Omega$ the mapping $i \mapsto f(i, \omega)$ will not be $\mathcal{I}$-measurable. Assume that f is essentially pairwise conditionally independent given $\mathcal{C}$, and also admits an essentially regular conditional distribution process μ given $\mathcal{C}$. By definition of the latter, we know that for λ-almost all $i \in I$, the marginal process $\Omega \in \omega \mapsto \mu_i(\omega) \in \Delta(T^0)$ is a regular conditional distribution of $\Omega \in \omega \mapsto f_i(\omega) \in T^0$ given $\mathcal{C}$.

Let $\bar{\mu} := \int_I \mu_i d\lambda$ be the mean conditional signal distribution over all agents. Then the Fubini property implies that $\bar{\mu}$ is a $\mathcal{C}$-measurable mapping from Ω to $\Delta(T^0)$. We assume that the process f is *non-trivial* in the sense that $\mathcal{C}$ is the same as the completed sub-σ-algebra of $\mathcal{F}$ generated by $\bar{\mu}$ together with the P-null subsets of Ω. This means that the mean conditional signal distribution carries the same information as the true state.

Let $\{A_s\}_{s \in S}$ be the $\mathcal{C}$-measurable partition of Ω such that $\tilde{s}(\omega) = s$ for any $\omega \in A_s$. Then $\mathcal{C}$ is generated by the finite family $\{A_s\}_{s \in S}$. Since $\bar{\mu}$ is $\mathcal{C}$-measurable, there exists a corresponding finite collection of measures $\{\mu_s\}_{s \in S}$ in $\Delta(T^0)$ such that $\bar{\mu}(\omega) = \sum_{s \in S} 1_{A_s}(\omega) \mu_s$ for P-almost all $\omega \in \Omega$. It is clear that μ_s is the agents' average signal distribution conditional on the true state being s. The non-triviality assumption above implies that

$$\forall s, s' \in S, \qquad s \neq s' \Longrightarrow \mu_s \neq \mu_{s'}. \tag{5}$$

5.2 A state contingent large economy

First, we define a complete information economy $\mathcal{E}^c$. The common **consumption set** of each agent $i \in I$ is the positive orthant $\mathbb{R}^m_+$. Suppose that for any given $i \in I$ and true state $s \in S$, the mapping $\mathbb{R}^m_+ \ni x \mapsto u_i(x; s)$ is agent i's **utility function** when the state is s. For any given $i \in I$ and $s \in S$, assume that $\mathbb{R}^m_+ \ni x \mapsto u_i(x; s) \in \mathbb{R}$ is continuous and **strictly monotonic** in $x \in \mathbb{R}^m_+$ in the sense that

$$\tilde{x} \geqq x \text{ and } \tilde{x} \neq x \Longrightarrow u_i(\tilde{x}; s) > u_i(x; s)$$

Assume too that for any fixed $x \in \mathbb{R}^m_+$ and $s \in S$, the mapping $I \ni i \mapsto u_i(x; s)$ is $\mathcal{I}$-measurable in $i \in I$.[5]

In this section, let $\|x\|$ denote the Euclidean norm of any vector $x \in \mathbb{R}^m$. Assume also that, in addition to continuity of each individual's utility function $\mathbb{R}^m_+ \ni x \mapsto u_i(x; s)$, the entire family of utility functions $\mathbb{R}^m_+ \ni x \mapsto u_i(x; s)$ as (i, s) varies over $I \times S$ is **uniformly equicontinuous** in the sense that, for any $\epsilon > 0$, there exists $\delta > 0$ such that $\|x - \tilde{x}\| < \delta$ implies $|u(i, x, s) - u(i, \tilde{x}, s)| < \epsilon$ for all $i \in I$, all $x, \tilde{x} \in \mathbb{R}^m_+$, and all $s \in S$.

Let $I \ni i \mapsto e(i) \in \mathbb{R}^m_+$ be the λ-integrable **endowment function** specifying each agent i's initial endowment. Assume that the *mean endowment vector* $\bar{e} := \int_I e(i)\, d\lambda$ satisfies $\bar{e} \gg 0$, meaning that the mean endowment of each good is positive. Let Δ_m denote the unit simplex in $\mathbb{R}^m_+$.

For each $s \in S$, the collection $\mathcal{E}^c_s = \{(I, \mathcal{I}, \lambda), u^I_s, e^I\}$, consisting of an atomless probability space of agents with their respective utility functions $x \mapsto u_i(x; s)$ and endowment vectors e_i, together constitutes a large deterministic exchange economy. A *complete information economy* is a collection $\mathcal{E}^c = \{\mathcal{E}^c_s : s \in S\}$ specifying the deterministic economy $\mathcal{E}^c_s$ for each $s \in S$. The following provides the definition of the basic concept of a Walrasian allocation.

Definition 4 1. *An* **allocation** *for $\mathcal{E}^c$ is a function*

$$I \times S \ni (i, s) \mapsto x^c_s(i) \in \mathbb{R}^m_+ \tag{6}$$

such that for any fixed $s \in S$, the mapping $i \mapsto x^c_s(i)$ is λ-integrable.

2. *An allocation $(i, s) \mapsto x^c_s(i)$ is* **feasible** *in $\mathcal{E}^c$ if, for each $s \in S$, one has $\int_I x^c_s(i)\, d\lambda = \int_I e(i)\, d\lambda$ (i.e., x^c_s is feasible in $\mathcal{E}^c_s$).*

3. *A feasible allocation $(i, s) \mapsto x^c_s(i)$ is a* **Walrasian** *(or competitive equilibrium)* **allocation** *in $\mathcal{E}^c$ if for each $s \in S$, there is a* **price system** *$p_s \in \Delta_m$ which, together with the*

[5] In the sequel, we shall often use subscripts to denote some argument of a function that is viewed as a parameter in a particular context.

feasible allocation x^c_s, makes (x^c_s, p_s) a competitive or Walrasian equilibrium in the large deterministic economy $\mathcal{E}^c_s$, in the sense that for λ-a.e. $i \in I$, given i's ***Walrasian budget set***

$$B_i(p_s) := \{x \in \mathbb{R}^m_+ : p_s \cdot x \leq p_s \cdot e(i)\} \tag{7}$$

one has

$$x^c_s(i) \in \arg\max_x \{u_i(x; s) : x \in B_i(p_s)\} \tag{8}$$

5.3 Monte Carlo sampling economies

We shall now apply Monte Carlo sampling to economies with a continuum of agents who have asymmetric information. Each agent $i \in I$ is informed about her private signal $t_i \in T^0$, but not the true state $s \in S$. Fix any $i^\infty \in I^\infty$ drawn from the iteratively completed infinite product probability space $(I^\infty, \bar{\mathcal{I}}^\infty, \bar{\lambda}^\infty)$. In the asymmetric information Monto Carlo sampling economy $\mathcal{E}^{i^\infty}$, there is a countable set of sampled agents $i^\infty \in I^\infty$.

For any $x \in \mathbb{R}^m_+$ and $t \in T$, let $U_i(x|t) := \sum_{s \in S} u_i(x; s) P^S(\{s\}|t)$ denote agent i's conditional expected utility of consumption bundle x given the type t.

A function z from $(T, \mathcal{T})$ to $\mathbb{R}^m_+$ is said to be a *consumption plan* in $\mathcal{E}^{i^\infty}$ if for any pair $t, t' \in T$ of type profiles that coincide on i^∞, one has $z(t) = z(t')$. That is, a consumption plan only depends on reported types of agents in the set $I(i^\infty)$ defined by

$$I(i^\infty) := \cup_{k=1}^\infty \{i_k\} \tag{9}$$

Let $CP(i^\infty)$ be the space of consumption plans in $\mathcal{E}^{i^\infty}$. For any agent $i \in I(i^\infty)$ and a consumption plan $z \in CP(i^\infty)$, let

$$U_i(z) := \int_\Omega u_i(z(t); s)\, dP = \sum_{s \in S} \pi_s \int_T u_i(z(t); s)\, dP^T_s \tag{10}$$

be the overall expected utility of agent i for the consumption plan $t \mapsto z(t)$.

An allocation in $\mathcal{E}^{i^\infty}$ is a function $I^\infty \ni i^\infty \mapsto x^{i^\infty} \in CP(i^\infty)$. For any allocation x^{i^∞}, any agent $i \in I(i^\infty)$, and any pair of private signals $t_i, t'_i \in T^0$, let

$$U_i^{i^\infty}(x_i^{i^\infty}, t'_i|t_i) := \int_{S \times T_{-i}} u(i, x_i^{i^\infty}(t_{-i}, t'_i), s) dP^{S \times T_{-i}}(\cdot|t_i)$$

denote agent i's conditional expected utility when she receives the private signal t_i but misreports it as t'_i.

5.4 Asymptotically feasible and Pareto efficient incentive compatible allocations

To discuss incentive compatibility, we invoke the revelation principle due to [6, 23], extended in an obvious way to a continuum of consumers. That is, we consider a **direct revelation**

mechanism in which reporting one's type truthfully is a Bayesian equilibrium for every agent in the corresponding game of incomplete information. Specifically, let g denote the agents' joint reporting process $I \times T \ni (i,t) \mapsto g(i,t) \in T^0$ with $g(i,t) = \tilde{t}(i)$ for all $(i,t) \in I \times T$. Let $G : I^\infty \times T \to (T^0)^\infty$ be the corresponding Monte Carlo sampling process of g. The following claim, which will be proved in Section 6.4, shows that g also has a stochastic macro structure.

Claim 2 *There is a countably generated sub-sigma-algebra $\mathcal{C}'$ of $\mathcal{T}$ such that g is regularly conditionally independent given $\mathcal{C}'$.*

By Theorem 1, this implies that the space $(I^\infty \times T, \bar{\mathcal{I}}^\infty \otimes \mathcal{T}, \bar{\lambda}^\infty \otimes P^T)$ has a one-way Fubini extension $(I^\infty \times T, \tilde{\mathcal{W}}, \tilde{Q})$ such that G is $\tilde{\mathcal{W}}$-measurable.

Definition 5.1 *1. An **allocation mechanism** is a mapping*

$$I^\infty \times I \times T \ni (i^\infty, i, t) \mapsto x^{i^\infty}(i,t) \in \mathbb{R}^m_+$$

2. *The allocation mechanism $(i^\infty, i, t) \mapsto x^{i^\infty}(i,t)$ is* asymptotically feasible *if, for $\tilde{Q}$-almost all $(i^\infty, t) \in I^\infty \times T$, one has*

$$\left\| \frac{1}{n} \sum_{k=1}^{n} x^{i^\infty}(i_k, t) - \frac{1}{n} \sum_{k=1}^{n} e(i_k) \right\| \to 0 \text{ as } n \to \infty$$

3. *The allocation mechanism $(i^\infty, i, t) \mapsto x^{i^\infty}(i,t)$ is* incentive compatible *if, for $\tilde{Q}$-almost all $(i^\infty, t) \in I^\infty \times T$, the* incentive constraint $U_i^{i^\infty}\left(x_i^{i^\infty}, t|t\right) \geq U_i^{i^\infty}\left(x_i^{i^\infty}, t'|t\right)$ *holds for any $i \in I(i^\infty)$ and any $t' \in T^0$.*

4. *The allocation mechanism $(i^\infty, i, t) \mapsto x^{i^\infty}(i,t)$ is* ex post individually rational *if, for $\tilde{Q}$-almost all $(i^\infty, t) \in I^\infty \times T$, one has $U_i^{i^\infty}\left(x_i^{i^\infty}|t\right) \geq U_i^{i^\infty}(e_i|t)$.*

5. *The allocation mechanism $(i^\infty, i, t) \mapsto x^{i^\infty}(i,t)$ is* asymptotically Pareto efficient *if, for $\tilde{Q}$-almost all $(i^\infty, t) \in I^\infty \times T$, the following holds: for any $\epsilon > 0$, there is no sequence $y : \mathbb{N} \to \mathbb{R}^m_+$ such that: (i) as $n \to \infty$, so $\left\|\frac{1}{n}\sum_{k=1}^{n} y_k - \frac{1}{n}\sum_{k=1}^{n} e(i_k)\right\| \to 0$; (ii) for any $i \in I(i^\infty)$, one has $U_i^{i^\infty}(y_i|t) \geq U_i^{i^\infty}(x_i^{i^\infty}|t) + \epsilon$.*

Now we are ready to state the following result for economies generated by Monto Carlo sampling.

Theorem 3 *There exists an allocation mechanism $(i^\infty, i, t) \mapsto x^{i^\infty}(i,t)$ which is asymptotically feasible, incentive compatible, ex post individually rational and asymptotically Pareto efficient.*

6 Appendix

6.1 Iteratively complete products

Let $(I_k, \mathcal{I}_k, \lambda_k)(k \in \mathbb{N})$ be a sequence of probability spaces. We use the same notation whether or not the spaces $\mathcal{P}_k = (I_k, \mathcal{I}_k, \lambda_k)$ are identical copies of a fixed space $(I, \mathcal{I}, \lambda)$. Let

$$\mathcal{P}^n := \prod_{k=1}^{n} \mathcal{P}_k = (I^n, \mathcal{I}^n, \lambda^n) := \left(\prod_{k=1}^{n} I_k,\ \bigotimes_{k=1}^{n} \mathcal{I}_k,\ \bigotimes_{k=1}^{n} \lambda_k \right)$$

denote the respective n-fold product, and let

$$P^\infty := \prod_{k=1}^{\infty} \mathcal{P}_k = (I^\infty, \mathcal{I}^\infty, \lambda^\infty) := \left(\prod_{k=1}^{\infty} I_k,\ \bigotimes_{k=1}^{\infty} \mathcal{I}_k,\ \bigotimes_{k=1}^{\infty} \lambda_k \right)$$

denote the infinite product counterpart.

The following definition is taken from [12].

Definition 5 *A subset E of the n-fold Cartesian product set I^n is said to be* ***iteratively null*** *in $\mathcal{P}^n$ if for every permutation π on $\{1, \ldots, n\}$, the n-fold iterated integral*

$$\int_{i_{\pi(1)} \in I_{\pi(1)}} \cdots \int_{i_{\pi(n)} \in I_{\pi(n)}} 1_E(i_1, i_2, \ldots, i_n)\ d\lambda_{\pi(n)}(i_{\pi(n)}) \ldots d\lambda_{\pi(1)}(i_{\pi(1)}) \tag{11}$$

of the indicator function $I^n \ni i^n \mapsto 1_E(i^n) \in \{0,1\}$ for the set E is well-defined and has value zero; in other words, for $\lambda_{\pi(1)}$-a.e. $i_{\pi(1)} \in I_{\pi(1)}$, $\lambda_{\pi(2)}$-a.e. $i_{\pi(2)} \in I_{\pi(2)}$, $\ldots$, $\lambda_{\pi(n)}$-a.e. $i_{\pi(n)} \in I_{\pi(n)}$, one has $(i_1, i_2, \ldots, i_n) \notin E$.

The following two propositions from [12] show that one can extend both the finite product probability space $\mathcal{P}^n$ and the infinite product probability space $\mathcal{P}^\infty$ by including all iteratively null sets, then forming the iterated completion.

Proposition 3 *Given any $n \in \mathbb{N}$, let $\mathcal{E}_n$ denote the family of all iteratively null sets in the n-fold product $(I^n, \mathcal{I}^n, \lambda^n)$. Then there exists a complete and countably additive probability space*

$$\bar{\mathcal{P}}^n := (I^n, \bar{\mathcal{I}}^n, \bar{\lambda}^n) := \left(I^n,\ \overline{\bigotimes}_{k=1}^{n} \mathcal{I}_k,\ \overline{\bigotimes}_{k=1}^{n} \lambda_k \right)$$

that satisfies the Fubini property, with:

1. *$\bar{\mathcal{I}}^n$ as the σ-algebra $\sigma(\mathcal{I}^n \cup \mathcal{E}_n)$, which is equal to the collection*

$$\mathcal{I}^n \triangle \mathcal{E}_n := \{\, D \triangle E : D \in \mathcal{I}^n,\ E \in \mathcal{E}_n \,\};$$

2. *$\bar{\lambda}^n$ as the unique measure that satisfies $\bar{\lambda}^n(D \triangle E) := \lambda^n(D)$ whenever $D \in \mathcal{I}^n$ and $E \in \mathcal{E}_n$.*

Proposition 4 *There exists a countably additive probability space*

$$\bar{\mathcal{P}}^\infty := (I^\infty, \bar{\mathcal{I}}^\infty, \bar{\lambda}^\infty) := \left(I^\infty,\ \overline{\bigotimes}_{k=1}^{\infty} \mathcal{I}_k,\ \overline{\bigotimes}_{k=1}^{\infty} \lambda_k\right)$$

in which

1. *$\bar{\mathcal{I}}^\infty$ is the σ-algebra generated by the union $\mathcal{G} := \cup_{n=1}^\infty \mathcal{G}_n$ of the families $\mathcal{G}_n$ of cylinder sets taking the form $G_n = A \times \prod_{k=n+1}^\infty I_k$ for some $A \in \bar{\mathcal{I}}^n$;*

2. *$\bar{\lambda}^\infty$ is the unique countably additive extension to $\bar{\mathcal{I}}^\infty$ of the set function $\mu : \mathcal{G} \to [0,1]$ defined so that $\mu(A \times \prod_{k=n+1}^\infty I_k) := \bar{\lambda}^n(A)$ for all $A \in \bar{\mathcal{I}}^n$.*

Moreover, for any $\bar{D} \in \bar{\mathcal{I}}^\infty$, there exist $D \in \mathcal{I}^\infty$ and $E \in \bar{\mathcal{I}}^\infty$ such that $\bar{D} = D \triangle E$ and $\bar{\lambda}^\infty(E) = 0$.

Unlike the finite product $\bar{\mathcal{P}}^n$, the infinite product measure space $\bar{\mathcal{P}}^\infty$ in Proposition 4 may not be complete in the usual sense. One can always complete it by the usual procedure (see, for example, [9] pp. 78–79). We still use the same notation to denote this completion, which also retains the property stated in the last sentence of Proposition 4.

The completed probability space $\bar{\mathcal{P}}^\infty$ will be called the *iterated completion* of $\mathcal{P}^\infty$, as well as the *iteratively complete product* of the spaces $\mathcal{P}_k$ ($k \in \mathbb{N}$). Let $i^\infty = (i_1, i_2, \ldots, i_n, \ldots)$ denote a general element of I^∞.

6.2 Proof of Theorem 1

First, let

$$D^\infty := \{i^\infty = (i_k)_{k=1}^\infty \in I^\infty : (g_{i_k})_{k=1}^\infty \text{ is mutually conditionally independent given } \mathcal{C}\} \quad (12)$$

denote the set of all infinite sequences $i^\infty \in I^\infty$ such that the associated sequence of random variables g_{i_k} ($k \in \mathbb{N}$) are mutually conditionally independent given $\mathcal{C}$.

Next, for any $n \in \mathbb{N}$, let

$$D^n := \{i^n = \{i_k\}_{k=1}^n \in I^n : \{g_{i_k}\}_{k=1}^n \text{ is mutually conditionally independent given } \mathcal{C}\}$$

denote the projection of the set $D^\infty \subset I^\infty$ onto the finite subproduct set I^n of all sequences of length n. Since g is essentially pairwise conditionally independent given $\mathcal{C}$, and also admits an essentially regular conditional distribution process μ given $\mathcal{C}$, Theorem 1 in [12] implies that $D^n \in \bar{\mathcal{I}}^n$ and $\bar{\lambda}^n(D^n) = 1$ for any $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, let

$$E^n := \{(i^\infty, j^\infty) \in I^\infty \times I^\infty : (i_1, \ldots, i_n, j_1, \ldots, j_n) \in D^{2n}\}$$

It is easy to see that $\overline{\bar{\lambda}^\infty \otimes \bar{\lambda}^\infty}(E^n) = \bar{\lambda}^{2n}(D^{2n}) = 1$. Let $E = \cap_{n=1}^\infty E^n$. It is clear that

$$\overline{\bar{\lambda}^\infty \otimes \bar{\lambda}^\infty}(E) = 1 \tag{13}$$

Also, for any $(i^\infty, j^\infty) \in E$, let $G(i^\infty) := \{g_{i_k} : k \in \mathbb{N}\}$ and $G(j^\infty) := \{g_{j_k} : k \in \mathbb{N}\}$ denote the associated countable sets of random variables. Then we know that the random variables in the set $G(i^\infty) \cup G(j^\infty)$ are mutually conditionally independent given $\mathcal{C}$. It follows from (13) that, for $\bar{\lambda}^\infty$-a.e. $i^\infty \in I^\infty$, the random variables in $G(i^\infty) \cup G(j^\infty)$ are mutually conditionally independent given $\mathcal{C}$ for $\bar{\lambda}^\infty$-a.e. $j^\infty \in I^\infty$.

Note that the infinite product σ-algebra $\mathcal{B}^\infty$ is generated by the family of all infinite cylinder sets which, for some $n \in \mathbb{N}$ and some collection $B_1, \dots B_n \in \mathcal{B}$ of n Borel sets, take the form $\prod_{i=1}^n B_i \times X^\infty$. To prove that $\bar{\mu}$ is an essentially regular conditional distribution process given $\mathcal{C}$, it is enough to consider the π-system consisting of these cylinder sets.

Fix any $i^\infty = (i_k)_{k=1}^\infty \in D^\infty$, where D^∞ was defined in (12). For any $B_1, \dots B_n \in \mathcal{B}$, mutual conditional independence given $\mathcal{C}$ of all the random variables in the sequence $(g_{i_k})_{k=1}^\infty$ implies that for P-a.e. $\omega \in \Omega$ one has

$$\begin{aligned}
P\left((G(i^\infty))^{-1} (B_1 \times \cdots \times B_n \times X^\infty) \,|\, \mathcal{C} \right)(\omega) &= P\left((g_{i_1}, \dots, g_{i_n})^{-1} (B_1 \times \cdots \times B_n) \,|\, \mathcal{C} \right)(\omega) \\
&= P\left(g_{i_1}^{-1}(B_1) \,|\, \mathcal{C} \right)(\omega) \cdots P\left(g_{i_n}^{-1}(B_n) \,|\, \mathcal{C} \right)(\omega) \\
&= \mu_{i_1\omega}(B_1) \cdots \mu_{i_n\omega}(B_n)
\end{aligned}$$

But definition (3) implies that

$$\mu_{i_1\omega}(B_1) \cdots \mu_{i_n\omega}(B_n) = \bar{\mu}_{i^\infty\omega}(B_1 \times \cdots \times B_n \times X^\infty)$$

So this proves that $I^\infty \times \Omega \ni (i^\infty, \omega) \mapsto \bar{\mu}_{i^\infty\omega}$ is an essentially regular conditional distribution process of G given $\mathcal{C}$. Therefore, Theorem 1 in [14] implies that there exists a one-way Fubini extension $(I^\infty \times \Omega, \tilde{\mathcal{W}}, \tilde{Q})$ of $(I^\infty \times \Omega, \bar{\mathcal{I}}^\infty \otimes \mathcal{F}, \bar{\lambda}^\infty \otimes P)$ such that G is $\tilde{\mathcal{W}}$-measurable.

6.3 Proofs of the results in Section 4

Proof of Proposition 1:

Take as given the real-valued functions h and f specified at the start of Section 4, as well as the regular conditional process $I \times \Omega \ni (i, \omega) \mapsto \mu_{i\omega} \in \mathcal{M}(X)$ defined in Section 3. For any $i \in I$ and $\omega \in \Omega$, let

$$\varphi(i, \omega) := \int_X h_i(x)\, d\mu_{i\omega} \quad \text{and} \quad \psi(i, \omega) := f(i, \omega) - \varphi(i, \omega) \tag{14}$$

We first prove that the random variable $\Omega \ni \omega \mapsto \int_I \varphi(i, \omega)\, d\lambda = \int_I \left[\int_X h_i(x)\, d\mu_{i\omega} \right] d\lambda$ belongs to $L^2(P)$. The property of essentially regular conditional distribution processes implies that

$$\text{for } \lambda\text{-almost all } i \in I, \text{ one has } \varphi(i, \omega) = \mathbb{E}[f(i)|\mathcal{C}](\omega) \ P\text{-a.s.} \tag{15}$$

Thus, by the Fubini property and Jensen's inequality, one has

$$\begin{array}{rlcl} & \int_\Omega \left[\int_I \varphi(i,\omega)\, d\lambda\right]^2 dP & = & \int_\Omega \left[\int_I \mathbb{E}[f(i)|\mathcal{C}](\omega)\, d\lambda\right]^2 dP \\ \leq & \int_\Omega \left[\int_I (\mathbb{E}[f(i)|\mathcal{C}](\omega))^2\, d\lambda\right] dP & = & \int_I \left[\int_\Omega (\mathbb{E}[f(i)|\mathcal{C}](\omega))^2\, dP\right] d\lambda \\ \leq & \int_I \left[\int_\Omega \mathbb{E}[f^2(i)|\mathcal{C}](\omega) dP\right] d\lambda & = & \int_I \left[\int_\Omega f^2(i,\omega)\, dP\right] d\lambda \end{array}$$

Because of our assumption that $\int_I \left[\int_\Omega h_i^2(g_i(\omega))\, dP\right] d\lambda = \int_I \left[\int_\Omega f_i^2(\omega)\, dP\right] d\lambda$ is finite, the last integral is finite. This proves that the function $\omega \mapsto \int_I \varphi(i,\omega)\, d\lambda$ also belongs to $L^2(P)$. Also φ can be viewed as essentially a function from $(I, \mathcal{I}, \lambda)$ to $L^2(\mathcal{C}, P)$, the space of real-valued, $\mathcal{C}$-measurable and square integrable functions on $(\Omega, \mathcal{F}, P)$.

Since $\mathcal{C}$ is countably generated, we know that $L^2(\mathcal{C}, P)$ is separable, which implies that φ is λ-essentially separably valued.[6] It is easy to see that φ is also weakly λ-measurable.[7] Then Theorem 2 in page 42 of [7] implies that the function $i \mapsto \varphi(i)$ is λ-measurable. Hence, there exists a sequence of simple functions $i \mapsto \varphi^k(i)$ with $\lim_{k\to\infty} \|\varphi^k - \varphi\| = 0$ for λ-a.e. $i \in I$. From (15) note that $\int_I \|\varphi(i)\|^2 d\lambda = \int_I \left[\int_\Omega (\mathbb{E}[f(i)|\mathcal{C}](\omega))^2\, dP\right] d\lambda$. Thus, Jensen's inequality implies that

$$\int_I ||\varphi(i)||^2 d\lambda \leq \int_I \int_\Omega \mathbb{E}[f^2(i)|\mathcal{C}]\, dP d\lambda = \int_I \int_\Omega f(i)^2 dP d\lambda < \infty$$

But then the Cauchy–Schwarz inequlity implies that

$$\int_I \|\varphi(i)\| d\lambda \leq \left(\int_I \|\varphi(i)\|^2 d\lambda\right)^{\frac{1}{2}} < \infty$$

By Theorem 2 on page 45 of [7], we know that φ viewed as a function from $(I, \mathcal{I}, \lambda)$ to $L^2(P)$ is Bochner integrable. Next, the classical law of large numbers for Bochner integrable functions, as shown in [3] and [22] (see also [8] and [17]), says that for λ^∞-a.e. $i^\infty \in I^\infty$, one has

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=1}^n \varphi(i_k) - \int_I \int_X h_i(x)\, d\mu_{i\omega} d\lambda \right\| = 0 \tag{16}$$

The proof of Lemma 11 in [13] shows that there exists $D^* \in \bar{\mathcal{I}}^\infty$ with $\bar{\lambda}^\infty(D^*) = 1$ such that for any $i^\infty \in D^*$, the random variables $(\psi_{i_k})_{k=1}^\infty$ defined by (14) are mutually orthogonal. This implies that for any $i^\infty \in D^*$, we have

$$\left\| \frac{1}{n} \sum_{k=1}^n \psi(i_k) \right\|^2 = \frac{1}{n^2} \sum_{k=1}^n \|\psi(i_k)\|^2 \tag{17}$$

Since $\int_I \|\psi(i)\|^2\, d\lambda < \infty$, the usual strong law of large numbers implies that for λ^∞-a.e. $i^\infty \in I^\infty$ one has

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \|\psi(i_k)\|^2 = \int_I \|\psi(i)\|^2\, d\lambda \tag{18}$$

[6]See page 42 in [7] for formal definitions.
[7]See page 41 in [7] for formal definitions.

It clearly follows that for $\bar{\lambda}^{\infty}$-a.e. $i^{\infty} \in I^{\infty}$, one has

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=1}^{n} \psi(i_k) \right\|^2 = 0 \tag{19}$$

Combining Equations (16) and (19), while using definition (14) of the function $i \mapsto \psi(i) \in \mathbb{R}$, it follows that for $\bar{\lambda}^{\infty}$-a.e. $i^{\infty} \in I^{\infty}$, one has

$$\begin{aligned} & \lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=1}^{n} f(i_k) - \int_I \int_X h_i(x)\, d\mu_{i\omega}(x)\, d\lambda \right\| \\ = \quad & \lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=1}^{n} \psi(i_k) + \frac{1}{n} \sum_{k=1}^{n} \varphi(i_k) - \int_I \int_X h_i(x)\, d\mu_{i\omega}(x) d\lambda \right\| = 0 \end{aligned}$$

This completes the proof. ■

The following lemma is a special case of a result in [13], which generalizes part of Lemma 2.1 in [17, p. 304] to the setting of iteratively complete product spaces.

Lemma 1 *For each $n \in \mathbb{N}$, let S_n be a subset of I whose λ-outer measure is one. Then the $\bar{\lambda}^{\infty}$-outer measure of $\prod_{n=1}^{\infty} S_n$ is also one.*

The next lemma is also taken from [13]. It generalizes to iteratively complete products one part of Theorem 2.4 in [17, p. 310], which is due to Talagrand.

Lemma 2 *Let g be a real-valued function on $(I, \mathcal{I}, \lambda)$. Suppose there is a real constant c such that*

$$\lim_{n\to\infty} \frac{g(i_1) + \cdots + g(i_n)}{n} = c \quad \text{for } \bar{\lambda}^{\infty}\text{-a.e. } i^{\infty} \in I^{\infty} \tag{20}$$

Then g is λ-integrable, with $\int_I g(i)\, d\lambda = c$.

The proof of following lemma adapts some of the ideas used in the proofs of Lemma 2.1 and Theorem 2.4 in [17], and of Lemma 1.4 in [16].

Lemma 3 *If a function f from T to a Banach space $\mathbb{B}$ satisfies SLLN, then it is norm integrably bounded.*

Proof: Let $f \in \text{SLLN}(\mathbb{B})$, with $\|a - \frac{1}{n}\sum_{k=1}^{n} f(i_k)\| \to 0$ for $\bar{\lambda}^{\infty}$-a.e. $i^{\infty} \in I^{\infty}$. Let D be the set of all $i^{\infty} \in I^{\infty}$ such that $\|\frac{1}{n} f(i_n)\| \to 0$ as $n \to \infty$. Because of the decomposition

$$\frac{1}{n} f(i_n) = -\left[a - \frac{1}{n} \sum_{k=1}^{n} f(i_k) \right] + \frac{n-1}{n} \left[a - \frac{1}{n-1} \sum_{k=1}^{n-1} f(i_k) \right] + \frac{1}{n} a$$

it follows that

$$\left\|\frac{1}{n}f(i_n)\right\| \le \left\|a - \frac{1}{n}\sum_{k=1}^{n} f(i_k)\right\| + \frac{n-1}{n}\left\|a - \frac{1}{n-1}\sum_{k=1}^{n-1} f(i_k)\right\| + \frac{1}{n}\|a\| \tag{21}$$

Now each term on the right-hand side of (21) converges $\bar{\lambda}^\infty$-a.s. to 0, so $\bar{\lambda}^\infty(D) = 1$.

Let $i \mapsto g(i)$ be an **upper λ-envelope** of $i \mapsto \|f(i)\|$, in the sense that $g : I \to \mathbb{R}_+ \cup \{\infty\}$ is an $\mathcal{I}$-measurable function satisfying: (i) $g(i) \ge \|f(i)\|$ for all $i \in I$; (ii) for any $\mathcal{I}$-measurable function h from I to $\mathbb{R}_+ \cup \{\infty\}$, the λ-inner measure of the set $\{\, i \in I : \|f(i)\| \le h(i) < g(i) \,\}$ is zero (see [17, p. 302]). For each $n \in \mathbb{N}$, define

$$S_n := \{i \in I : g(i) \le 2\|f(i)\| \text{ or } \|f(i)\| \ge n\}$$

Define the function $h_n := \min\{n, \frac{1}{2}g\}$ on I, which is evidently $\mathcal{I}$-measurable. Also, it is clear that $\|f(i)\| < h_n(i) < g(i)$ for all $i \in I \setminus S_n$ (even when $g(i) = \infty$). By definition of the upper λ-envelope, therefore, the set $I \setminus S_n$ must have λ-inner measure zero, implying that its λ-outer measure of S_n is one. Lemma 1 says that then the set $\prod_{n=1}^\infty S_n$ also has $\bar{\lambda}^\infty$-outer measure one, and so therefore does $D \cap \prod_{n=1}^\infty S_n$.

Fix any $i^\infty \in D \cap \prod_{n=1}^\infty S_n$. Since $\|\frac{1}{n}f(i_n)\| \to 0$ as $n \to \infty$, one must have $\|f(i_n)\| < n$ for sufficiently large n, and then $i_n \in S_n$ implies that $0 \le g(i_n) \le 2\|f(i_n)\|$. Hence, $\frac{1}{n}g(i_n) \to 0$. But g is $\mathcal{I}$-measurable by definition, so $\frac{1}{n}g(i_n) \to 0$ for all i^∞ in some $\mathcal{I}^\infty$-measurable superset E of $D \cap \prod_{n=1}^\infty S_n$. Since the $\bar{\lambda}^\infty$-outer measure of $D \cap \prod_{n=1}^\infty S_n$ is one, it follows that $\bar{\lambda}^\infty(E) = \lambda^\infty(E) = 1$.

Given any $i^\infty \in I^\infty$, let $\phi(i^\infty) := \sup_{n\in\mathbb{N}} \frac{1}{n}g(i_n)$. Then $\phi(i^\infty)$ is finite for all $i^\infty \in E$. Because g is $\mathcal{I}$-measurable, the function $\phi : I \to \mathbb{R}_+ \cup \{\infty\}$ must be $\mathcal{I}^\infty$-measurable. So there exists a positive integer K such that

$$\lambda^\infty\left(\{\, i^\infty \in I^\infty : \phi\left(i^\infty\right) < K \,\}\right) > \tfrac{1}{2} \tag{22}$$

For each $n \in \mathbb{N}$, let $\alpha_n := \lambda\left(\{i \in I : g(i) \ge nK\}\right)$. Because λ^∞ is a product measure, it is evident that

$$\lambda^\infty\left(\{\, i^\infty \in I^\infty : \phi\left(i^\infty\right) < K \,\}\right) = \prod_{n=1}^{\infty}(1-\alpha_n) \tag{23}$$

Obviously (22) and (23) imply that $\prod_{n=1}^\infty(1-\alpha_n) > \frac{1}{2}$. But $\ln(1-\alpha_n) \le -\alpha_n$, so

$$\sum_{n=1}^{\infty}\alpha_n \le -\sum_{n=1}^{\infty}\ln(1-\alpha_n) < -\ln(1/2) = \ln 2 < \infty \tag{24}$$

This implies that $\lim_{n\to\infty}\alpha_n = 0$, and so $\lambda\left(\{i \in I : g(i) = \infty\}\right) = 0$.

Given any fixed $i \in I$ with $g(i) < \infty$, let m be the smallest integer such that $g(i) < mK$. Then $g(i) \in [nK, \infty)$ for $n \in \{1, \ldots, m-1\}$, and so $\sum_{n=1}^{\infty} 1_{[nK,\infty)}(g(i)) = m - 1$. It follows that

$$g(i) \leq K + K \sum_{n=1}^{\infty} 1_{[nK,\infty)}(g(i)) \tag{25}$$

for all $i \in I$ with $g(i) < \infty$. Because $\lambda(\{i \in I : g(i) = \infty\}) = 0$, the definition of α_n implies that $\int_I 1_{[nK,\infty)}(g(i))\, d\lambda = \alpha_n$. It follows from (24) and (25), therefore, that

$$\int_I g\, d\lambda \leq K + K \sum_{n=1}^{\infty} \alpha_n < K(1 + \ln 2) < \infty$$

Finally, let f^* be the function from I to $\mathbb{R}_+$ such that $f^*(i) = g(i)$ when $g(i) < \infty$ and $f^*(i) = 0$ when $g(i) = \infty$. Clearly f^* is a norm dominant λ-integrable function for $\|f\|$, so f is norm integrably bounded. ■

Two functions f and $\tilde{f}$ from $(I, \mathcal{I}, \lambda)$ to a Banach space $\mathbb{B}$ are said to be **scalarly equivalent** if, for any $b' \in \mathbb{B}'$, the corresponding real-valued functions $i \mapsto \langle f(i), b' \rangle$ and $i \mapsto \langle \tilde{f}(i), b' \rangle$ are equal for λ-a.e. $i \in I$.

Lemma 4 *Let $\mathbb{H}$ be a Hilbert space and f a function in $\mathcal{L}(\lambda, \mathbb{H})$ that is scalarly equivalent to the zero function. Then*

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=1}^{n} f(i_k) \right\| = 0 \quad \textit{for } \bar{\lambda}^{\infty}\textit{-a.e. } i^{\infty} \in I^{\infty} \tag{26}$$

Proof: Given $f \in \mathcal{L}(\lambda, \mathbb{H})$, let $g : I \to \mathbb{R}_+$ be a λ-integrable function that norm dominates f. For each $k \in \mathbb{N}$, let X_k be the random variable defined on $(I^{\infty}, \mathcal{I}^{\infty}, \lambda^{\infty})$ by $X_k(i^{\infty}) := [g(i_k)]^2$. Since $\mathbb{E}X_1^{1/2} < \infty$ and the variables X_k are i.i.d., the Marcinkiewicz–Zygmund Theorem for the case $p = 1/2$ and $c = 0$ (see [5, p. 125]) implies that $n^{-2} \sum_{k=1}^{n} X_k(i^{\infty}) \to 0$ for λ^{∞}-a.e. $i^{\infty} \in I^{\infty}$. Because the definition of g implies that $\|f(i)\| \leq g(i)$ for all $i \in I$, we have $\|f(i_k)\|^2 \leq [g(i_k)]^2 = X_k(i^{\infty})$ for all $k \in \mathbb{N}$. It follows that $n^{-2} \sum_{k=1}^{n} \|f(i_k)\|^2 \to 0$ for λ^{∞}-a.e. $i^{\infty} \in I^{\infty}$.

Next, we follow the idea behind some of the computations in the proof of Theorem 1.3 in [8, p. 277]. For any $i^{\infty} \in I^{\infty}$, we have

$$\left\| \frac{1}{n} \sum_{k=1}^{n} f(i_k) \right\|^2 = \frac{1}{n^2} \sum_{k=1}^{n} \|f(i_k)\|^2 + \frac{2}{n^2} \sum_{1 \leq j < k \leq n} \langle f(i_j), f(i_k) \rangle \tag{27}$$

Because f is scalarly equivalent to zero, for any $h \in \mathbb{H}$ one has $\langle f(i), h \rangle = 0$ for λ-a.e. $i \in I$. In particular, for any $i' \in I$, one has $\langle f(i), f(i') \rangle = 0$ for λ-a.e. $i \in I$. Hence there exists a $\bar{\mathcal{I}}^2$-measurable set $D \subseteq I \times I$ such that $\bar{\lambda}^2(D) = 1$ and $\langle f(i), f(i') \rangle = 0$ for all $(i, i') \in D$. For each pair $j, k \in \mathbb{N}$, let D_{jk} denote the set of all sequences $i^\infty \in I^\infty$ such that $(i_j, i_k) \in D$, and define $D^* := \cap_{j=1}^\infty \cap_{k=j+1}^\infty D_{jk}$. Then for all $i^\infty \in D^*$ one has $\langle f(i_j), f(i_k)\rangle = 0$ for all $j, k \in \mathbb{N}$ with $j < k$. Obviously $D_{jk} \in \bar{\mathcal{I}}^\infty$ and $\bar{\lambda}^\infty(D_{jk}) = 1$ for each $j, k \in \mathbb{N}$, so $D^* \in \bar{\mathcal{I}}^\infty$ and $\bar{\lambda}^\infty(D^*) = 1$ also.

Combining the results in the last two paragraphs shows that (27) implies (26). ■

Proof of Theorem 2:

Let h be a function from I to the Banach space $\mathbb{B}$ such that, for some $a \in \mathbb{B}$, one has $\lim_{n\to\infty} \|a - \frac{1}{n}\sum_{k=1}^n h(i_k)\| = 0$ for λ^∞-a.e. $i^\infty \in I^\infty$. Take any fixed $b' \in \mathbb{B}'$, and let $I \ni i \mapsto g(i) \in \mathbb{R}$ be defined so that $g(i) := \langle h(i), b'\rangle$ for all $i \in I$. A routine calculation shows that, for $\bar{\lambda}^\infty$-a.e. $i^\infty \in I^\infty$, one has

$$\lim_{n\to\infty} \frac{1}{n}[g(i_1) + \cdots + g(i_n)] = \langle a, b'\rangle$$

Then Lemma 2 implies that g is λ-integrable, with $\int_I g(i)\, d\lambda = \langle a, b'\rangle$. Hence, h is Gel′fand integrable and has a as its Gel′fand integral. Lemma 3 implies that h is also norm integrably bounded.

Now suppose that $\mathbb{B}$ is a Hilbert space $\mathbb{H}$. Let f be any function in $\mathcal{L}(\lambda, \mathbb{H})$. Since f is Gel′fand integrable, it follows from [1, Theorem 11.51] (or [7, p. 52]) that for each $S \in \mathcal{I}$, the function $i \mapsto 1_S(i)\, f(i)$ is Gel′fand integrable, where $i \mapsto 1_S(i) \in \{0, 1\}$ is the indicator function of the measurable set S. Let $\nu(S)$ denote its Gel′fand integral over I, which is an element of $\mathbb{H}$. It follows that $\|\nu(S)\|^2 = \langle \nu(S), \nu(S)\rangle = \int_I \langle (1_S f)(i), \nu(S)\rangle d\lambda$. By the hypothesis of norm integrable boundedness, there exists a λ-integrable function $f^* : I \to \mathbb{R}_+$ such that $\|f(i)\| \le f^*(i)$ for λ-a.e. $i \in I$, and so $\langle (1_S f)(i), \nu(S)\rangle \le (1_S f^*)(i)\|\nu(S)\|$. Hence $\|\nu(S)\|^2 \le \int_I (1_S f^*)(i)\|\nu(S)\| d\lambda$. So even when $\nu(S) = 0$, one has

$$\|\nu(S)\| \le \int_S f^*(i)\, d\lambda \tag{28}$$

Let $S_1, S_2, \ldots \in \mathcal{T}$ be any countable collection of pairwise disjoint measurable subsets of T. Obviously $\nu\left(\cup_{k=1}^n S_k\right) = \sum_{k=1}^n \nu(S_k)$ for $n = 1, 2, \ldots$. Furthermore, (28) implies that

$$\sum_{k=1}^n \|\nu(S_k)\| \le \sum_{k=1}^n \int_{S_k} f^*(i)\, d\lambda \le \int_T f^*(i)\, d\lambda < +\infty \tag{29}$$

It follows that the sequence defined by $s_n := \nu\left(\cup_{k=1}^n S_k\right) = \sum_{k=1}^n \nu(S_k)$ is a Cauchy sequence, and so convergent in the complete normed space $\mathbb{H}$. Hence $\nu\left(\cup_{k=1}^\infty S_k\right) = \sum_{k=1}^\infty \nu(S_k)$. It follows from (29) that ν is an $\mathbb{H}$-valued σ-additive measure of bounded variation. Moreover, (28) also implies that the vector measure ν is absolutely continuous w.r.t. λ.

Next, we shall show that f is scalarly equivalent to a Bochner integrable function ϕ from $(I, \mathcal{I}, \lambda)$ to $\mathbb{H}$. Because the Hilbert space $\mathbb{H}$ is a particular kind of reflexive Banach space, it has the Radon–Nikodym property (see [7, p. 82]). So there exists a Bochner integrable function ϕ from $(I, \mathcal{I}, \lambda)$ to $\mathbb{H}$ such that $\nu(S)$ equals the Bochner integral $\int_S \phi(i)\, d\lambda$ for each $S \in \mathcal{I}$. Now the Bochner integral, when it exists, must equal the Gel′fand integral (see, for example, [1, p. 423]). So given any $h \in \mathbb{H}$, it follows that

$$\langle \nu(S), h\rangle = \int_S \langle \phi(i), h\rangle d\lambda = \int_S \langle f(i), h\rangle d\lambda$$

Because the choice of $S \in \mathcal{I}$ was arbitrary, one has $\langle f(i), h\rangle = \langle \phi(i), h\rangle$ for λ-a.e. $i \in I$. That is, f is scalarly equivalent to ϕ.[8]

Define $\psi := f - \phi$. Because ϕ is Bochner integrable, it follows from [7, p. 45], for example, that $\|\phi\|$ is integrable. Clearly, then, ψ is norm integrably bounded, Gel′fand integrable, and scalarly equivalent to zero. So Lemma 4 implies that $\psi \in \mathrm{LLN}(\mathbb{H})$. Then the classical law of large numbers for Bochner integrable functions, as shown in [3] and [22] (see also [8] and [17]), says that ϕ is in $\mathrm{LLN}(\mathbb{H})$, and so in $\mathrm{SLLN}(\mathbb{H})$ as well. Therefore $f = \phi + \psi \in \mathrm{SLLN}(\mathbb{H})$. ■

Proof of Claim 1: Let φ be any square integrable random variable on $(\Omega, \mathcal{F}, P)$. By the property of regular conditional distribution process μ and the Fubini property, one has

$$\int_\Omega \left[\int_I \int_X h(i,x)\, d\mu_{i\omega} d\lambda\right] \varphi(\omega)\, dP = \int_I \int_\Omega \mathbb{E}[f(i,\omega)|\mathcal{C}]\varphi(\omega)\, dP d\lambda$$

Proposition 2 in [14] implies that for λ-almost all $i \in I$, the random variables $\omega \mapsto f_i(\omega)$ and $\omega \mapsto \varphi(\omega)$ are conditionally independent given $\mathcal{C}$. Therefore, we have

$$\int_\Omega \left[\int_I \int_X h(i,x)\, d\mu_{i\omega} d\lambda\right] \varphi(\omega)\, dP = \int_I \int_\Omega \mathbb{E}[f(i,\omega)\varphi(\omega)|\mathcal{C}]\, dP d\lambda = \int_I \int_\Omega f(i,\omega)\varphi(\omega)\, dP d\lambda$$

This implies that f is Gel′fand integrable.

6.4 Proof of Theorem 3

By the usual existence result on Walrasian allocations in [2] and [15], we know that there exists a Walrasian equilibrium (x^c, p) for the economy $\mathcal{E}^c$. Because we assumed that the utility function $\mathbb{R}^m_+ \ni x \mapsto u_i(x; s)$ of each agent $i \in I$ is strictly monotonic, we know that for any $s \in S$, the Walrasian equilibrium price vector p_s is strictly positive.

Note that, by assumption, the private signal process $I \times \Omega \ni (i,\omega) \mapsto f(i,\omega) \in T^0$ that was introduced in Section 5.1 is essentially pairwise conditionally independent given $\mathcal{C}$ and admits an essentially regular conditional distribution process μ given $\mathcal{C}$. Then Proposition 5

[8]The argument used in this paragraph is essentially the same as the simple argument on [7, p. 89], where the case of norm bounded functions is considered. See also [18] for discussion and for many additional references concerning this scalar equivalence result.

in [13] implies that for $\bar{\lambda}^\infty$-a.e. $i^\infty \in I^\infty$, there exists $F \in \mathcal{F}$ with $P(F) = 1$ such that for any $\omega \in F$ and any $q \in T_0$, one has

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} 1_q(f(i_k, \omega)) = [\bar{\mu}(\omega)](q) \tag{30}$$

The usual strong law of large numbers implies that for λ^∞-a.e. $i^\infty \in I^\infty$ one has

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} e(i_k) = \int_I e(i)\, d\lambda \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_s^c(i_k) = \int_I x_s^c(i)\, d\lambda \text{ for all } s \in S \tag{31}$$

Let D be the set of $i^\infty \in I^\infty$ such that Equations (30) and (31) both hold. It is clear that $\bar{\lambda}^\infty(D) = 1$.

First, for any $i^\infty \notin D$ and $i \in I(i^\infty)$, construct $x^{i^\infty}(i,t) := e(i)$ for all $t \in T$. Also, for any $i^\infty \notin D$ and $i \in I(i^\infty)$, the definition of Walrasian equilibrium implies that the inequality $U_i^{i^\infty}\left(x_i^{i^\infty}, t_i | t_i\right) \geq U_i^{i^\infty}\left(x_i^{i^\infty}, t_i' | t_i\right)$ holds for any $t_i, t_i' \in T^0$.

Alternatively, consider any fixed $i^\infty \in D$. For any $n \in \mathbb{N}$, $t \in T$ and $q \in T_0$, let $\gamma_n^{i^\infty}(t,q) := \frac{1}{n}\sum_{k=1}^{\infty} 1_{\{q\}}(t_{i_k})$. This defines a mapping $T \ni t \mapsto \gamma_n^{i^\infty}(t) \in \Delta(T^0)$. For any $t \in T$, given the counting measure $\bar{\gamma}$ on the finite set T^0, let

$$\gamma^{i^\infty}(t) := \begin{cases} \lim_{n\to\infty} \gamma_n^{i^\infty}(t) & \text{if the limit exists} \\ \bar{\gamma} & \text{otherwise} \end{cases} \tag{32}$$

Next, define the sets

$$L_s^{i^\infty} := \{t \in T : \gamma^{i^\infty}(t) = \mu_s\} \text{ for all } s \in S, \text{ and } L_0^{i^\infty} := T \setminus \bigcup_{s\in S} L_s^{i^\infty} \tag{33}$$

Because (30) holds P-a.s., it follows that $P_s^T(L_s^{i^\infty}) = 1$.

Also, the non-triviality assumption implies that for any s, $s' \in S$ with $s \neq s'$, one has $L_s^{i^\infty} \cap L_{s'}^{i^\infty} = \emptyset$. Thus, the collection $\{L_0^{i^\infty}\} \cup \{L_s^{i^\infty} : s \in S\}$ forms a measurable partition of T.

The definition (32) of γ^{i^∞} obviously implies that for any $i \in I(i^\infty)$ one has $\gamma^{i^\infty}(t_{-i}, t_i) = \gamma^{i^\infty}(t_{-i}, t_i')$ for all $t_{-i} \in T_{-i}$ and all $t_i, t_i' \in T^0$. Hence, for any $i \in I(i^\infty)$, $t \in T$, $t_i' \in T^0$, and $s \in S$, one has

$$t \in L_s^{i^\infty} \iff \gamma^{i^\infty}(t) = \mu_s \iff \gamma^{i^\infty}(t_{-i}, t_i') = \mu_s \iff (t_{-i}, t_i') \in L_s^{i^\infty}. \tag{34}$$

Since $L_0^{i^\infty}$ equals $T \setminus \cup_{s\in S} L_s^{i^\infty}$, we also know that $t \in L_0^{i^\infty} \iff (t_{-i}, t_i') \in L_0^{i^\infty}$. Hence, for any $i \in I(i^\infty)$ we have $x^{i^\infty}(i,t) = x^{i^\infty}(i,(t_{-i}, t_i'))$ for all $t \in T$ and $t_i' \in T^0$. This trivially implies that for any $i \in I(i^\infty)$ and any $t_i, t_i' \in T^0$, the allocation $I \times T \ni (i,t) \mapsto x^{i^\infty}(i,t) \in \mathbb{R}_+^m$ satisfies the corresponding incentive constraint

$$U_i^{i^\infty}\left(x_i^{i^\infty}, t_i | t_i\right) \geq U_i^{i^\infty}\left(x_i^{i^\infty}, t_i' | t_i\right) \tag{35}$$

For each $s \in S$, let δ_s denote the Dirac measure on S that gives probability one to the point s and zero to all the other points of S. Define a function H from T to the space $\Delta(S)$ of all probability measures on the finite set S by letting

$$H(t) := \begin{cases} \delta_s & \text{for the unique } s \in S \text{ such that } t \in L_s^{i^\infty} \\ \delta_{s_1} & \text{for } t \in L_0^{i^\infty} \end{cases}$$

Then the same proof as in Lemma 3 of [24] shows that for each $t \in T$, the measure $H(t)$ is a version of $P^s(\cdot|t)$.

Now we are ready to prove Claim 2.

Proof of Claim 2: Fix any $i^\infty \in D$. Let $\mathcal{C}'$ be the σ-algebra generated by the finite family $\{L_s^{i^\infty} : s \in S\}$. Note that for any $s \in S$, one has

$$P(\tilde{t} \in L_s^{i^\infty}) = \sum_{s' \in S} \pi_{s'} P_{s'}^T(L_s^{i^\infty}) = \pi_s P_s^T(L_s^{i^\infty}) = \pi_s \tag{36}$$

and $$P(\tilde{s} = s | \tilde{t} \in L_s^{i^\infty}) = \frac{P(\tilde{s} = s, \tilde{t} \in L_s^{i^\infty})}{P^T(L_s^{i^\infty})} = \frac{\pi_s P(\tilde{t} \in L_s^{i^\infty} | \tilde{s} = s)}{\pi_s P_s^T(L_s^{i^\infty})} = 1. \tag{37}$$

Fix any $s \in S$, $q, q' \in T^0$ and $i, j \in I$ such that the random variables f_i and f_j from Ω to T^0 are conditionally independent given s. We know that

$$P^T(g_i = q, g_j = q' | L_s^{i^\infty}) = \frac{P(f_i = q, f_j = q', \tilde{t} \in L_s^{i^\infty})}{P^T(L_s^{i^\infty})} = \frac{\pi_s P(f_i = q, f_j = q', \tilde{t} \in L_s^{i^\infty} | \tilde{s} = s)}{\pi_s P_s^T(L_s^{i^\infty})}$$

Because $P_s^T(L_s^{i^\infty}) = 1$, whereas f_i and f_j are conditionally independent given s, we have

$$\begin{aligned} P^T(g_i = q, g_j = q' | L_s^{i^\infty}) &= P(f_i = q, f_j = q', \tilde{t} \in L_s^{i^\infty} | \tilde{s} = s) \\ &= P(f_i = q, f_j = q' | \tilde{s} = s) \\ &= P(f_i = q | \tilde{s} = s) \cdot P(f_j = q' | \tilde{s} = s) \\ &= P^T(g_i = q | L_s^{i^\infty}) \cdot P^T(g_j = q' | L_s^{i^\infty}) \end{aligned}$$

By Equations (36) and (37), we know that

$$\begin{aligned} & P(f_i = q | \tilde{s} = s) \cdot P(f_j = q' | \tilde{s} = s) \\ = \ & \frac{1}{\pi_s^2} P(f_i = q, \tilde{s} = s) \cdot P(f_j = q', \tilde{s} = s) \\ = \ & \frac{1}{\pi_s^2} P(\tilde{t} \in L_s^{i^\infty})^2 \cdot P(f_i = q, \tilde{s} = s | \tilde{t} \in L_s^{i^\infty}) \cdot P(f_j = q', \tilde{s} = s | \tilde{t} \in L_s^{i^\infty}) \\ = \ & P(f_i = q | \tilde{t} \in L_s^{i^\infty}) \cdot P(f_j = q' | \tilde{t} \in L_s^{i^\infty}) \\ = \ & P^T(g_i = q | L_s^{i^\infty}) \cdot P^T(g_j = q' | L_s^{i^\infty}) \end{aligned}$$

This implies that $P^T(g_i = q, g_j = q' | L_s^{i^\infty}) = P^T(g_i = q | L_s^{i^\infty}) \cdot P^T(g_j = q' | L_s^{i^\infty})$. Hence, g is essentially pairwise conditionally independent given $\mathcal{C}'$.

For any $i \in I$, $s \in S$ and $t \in L_s^{i^\infty}$, let ν_{it} denote $\mu_{i(s,t)}$, where μ is the essentially regular conditional distribution process of f given $\mathcal{C}$. It is clear that ν is an essentially regular conditional distribution process of g given $\mathcal{C}'$. Therefore, g is regularly conditionally independent given $\mathcal{C}'$.

This completes the proof of Claim 2. We now continue the proof of Theorem 3.

Let $E_s^{i^\infty}$ be the set of all $t \in L_s^{i^\infty}$ such that $P^T(\cdot|t) = \delta_s$. Clearly $P_s^T(E_s^{i^\infty}) = 1$ for any $t \in E_s^{i^\infty}$. Let $E^{i^\infty} := \cup_{s \in S} E_s^{i^\infty}$. Then

$$P^T(E^{i^\infty}) = \sum_{s' \in S} \pi_{s'} P_{s'}^T(\cup_{s \in S} E_s^{i^\infty}) = \sum_{s' \in S} \pi_{s'} P_{s'}^T(E_{s'}^{i^\infty}) = \sum_{s' \in S} \pi_{s'} = 1$$

Given the Walrasian equilibrium allocation $(i, s) \mapsto x_s^c(i)$ for the economy $\mathcal{E}^c$, as specified by (6), define a mapping x^{i^∞} from $I \times T$ to $\mathbb{R}_+^m$ by letting

$$x^{i^\infty}(i, t) := \begin{cases} x_s^c(i) & \text{for the unique } s \in S \text{ such that } t \in L_s^{i^\infty} \\ e(i) & \text{if } t \in L_0^{i^\infty} \end{cases} \tag{38}$$

It is clear that x^{i^∞} only depends on reports from agents $i \in I(i^\infty)$. Hence x^{i^∞} is an allocation in the economy $\mathcal{E}^{i^\infty}$.

Note that for any $s \in S$ the feasibility condition in Part 2 of Definition 4 implies that $\int_I x_s^c(i)\, d\lambda = \int_I e(i)\, d\lambda$, and also

$$\begin{aligned} \lim_{n\to\infty} \tfrac{1}{n} \textstyle\sum_{k=1}^n x^{i^\infty}(i_k, t) &= \lim_{n\to\infty} \tfrac{1}{n} \textstyle\sum_{k=1}^n e(i_k) &&= \textstyle\int_I e(i)\, d\lambda, \text{ if } t \in L_0^{i^\infty} \\ \lim_{n\to\infty} \tfrac{1}{n} \textstyle\sum_{k=1}^n x^{i^\infty}(i_k, t) &= \lim_{n\to\infty} \tfrac{1}{n} \textstyle\sum_{k=1}^n x_s^c(i_k) &&= \textstyle\int_I x_s^c(i)\, d\lambda, \text{ if } t \in L_s^{i^\infty} \end{aligned}$$

These last equalities imply that, for any $t \in T$, as $n \to \infty$, the allocation defined by (38) satisfies the asymptotic feasibility condition

$$\left\| \frac{1}{n} \sum_{k=1}^n [x^{i^\infty}(i_k, t) - e(i_k)] \right\| \to \int_I [x_s^c(i) - e(i)] d\lambda = 0 \tag{39}$$

Now fix any $s \in S$ and $t \in E_s^{i^\infty}$. Evidently definition (7) implies that for any $i \in I$ one has $e(i) \in B_i(p_s)$. Since $P^T(\cdot|t) = \delta_s$, it follows from (8) that for any $i \in I(i^\infty)$, one has

$$U_i^{i^\infty}(x^{i^\infty}(i, t)|t) = u_i(x_s^c(i); s) \geq u_i(e(i); s) = U_i^{i^\infty}(e(i)|t). \tag{40}$$

This proves ex post individual rationality.

Finally, fix any $\epsilon > 0$. By uniform equicontinuity of the family of utility functions $\mathbb{R}_+^m \ni x \mapsto u_i(x; s)$ (for $i \in I$ and $s \in S$), there exists $\delta > 0$ such that whenever $x, x' \in \mathbb{R}_+^m$ satisfy $\|x - x'\| < \delta$, then $|u_i(x; s) - u_i(x'; s)| < \epsilon$ for all $i \in I$ and $s \in S$.

Let $\bar{p}_s := \min_{j\in\{1,2,\ldots,m\}} p_{sj}$ and $\delta' := \frac{1}{2}\bar{p}_s\delta$. For any $i \in I$ and $s \in S$, let $M_s^i := p_s \cdot e(i)$ denote the value of agent i's endowment at the equilibrium price vector p_s that applies in the economy $\mathcal{E}_s^c$.

Fix any $i \in I(i^\infty)$ and $x \in B(p_s, M_s^i + \delta')$. Let $x' = \frac{M_s^i}{M_s^i+\delta'}x$. It is clear that $x' \in B(p_s, M_s^i)$. By the definition of $\bar{p}_s$ and δ', we have

$$
\begin{aligned}
\|x - x'\| &= \frac{\delta'}{M_s^i}\|x'\| \leq \frac{\delta'}{p_s \cdot x'(i)}\|x'\| \\
\leq \quad & \frac{\delta'}{\bar{p}_s \cdot \sum_{j=1}^m x'_j(i)}\|x'\| \leq \frac{\delta'}{\bar{p}_s \cdot \sum_{j=1}^m x'_j(i)} \sum_{j=1}^m x'_j(i) = \frac{1}{2}\delta < \delta
\end{aligned}
$$

This implies that $u_i(x; s) < u_i(x'; s) + \epsilon$. For any $i \in I(i^\infty)$ and any $x \in B(p_s, M_s^i + \delta')$, it follows that

$$U_i^{i^\infty}(x|t) < U_i^{i^\infty}(x_i^{i^\infty}|t) + \epsilon \tag{41}$$

Let $I(i^\infty) \ni i \mapsto y_i \to \mathbb{R}_+^m$ be any sequence such that $U_i^{i^\infty}(y_i|t) \geq U_i^{i^\infty}(x_i^{i^\infty}|t) + \epsilon$ for all $i \in I(i^\infty)$. From (41) it follows that $p_s \cdot y_i \geq M_s^i + \delta'$ for all $i \in I(i^\infty)$, which implies that $\frac{1}{n}\sum_{k=1}^n p_s \cdot [y_{i_k} - e(i_k)] \not\to 0$. It is clear then that no sequence $I(i^\infty) \ni i \mapsto y_i \to \mathbb{R}_+^m$ such that $U_i^{i^\infty}(y_i|t) \geq U_i^{i^\infty}(x_i^{i^\infty}|t) + \epsilon$ for all $i \in I(i^\infty)$ can satisfy the asymptotic feasibility condition (39).

Finally, note that $G : I^\infty \times T \to (T^0)^\infty$ is the Monte Carlo sampling process of g, and that $(I^\infty \times T, \tilde{\mathcal{W}}, \tilde{Q})$ is a one-way Fubini extension of $(I^\infty \times T, \bar{\mathcal{I}}^\infty \otimes \mathcal{T}, \bar{\lambda}^\infty \otimes P^T)$ such that G is $\tilde{\mathcal{W}}$-measurable. Within the framework of this one-way Fubini extension, the arguments in this section establish that the allocation mechanism $(i^\infty, i, t) \mapsto x^{i^\infty}(i, t)$ is incentive compatible, asymptotically feasible, ex post individually rational, and asymptotically Pareto efficient. ■

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